UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
BOOKSTACKS

Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

http://www.archive.org/details/adaptivelestimat1372port

STX
330
B385
No. 1372 COPY 2

BEBR
FACULTY WORKING
PAPER NO. 1372

THE LIBRARY OF THE
SEP 24 1981
UNIVERSITY OF ILLINOIS
URBANA

Adaptive L-Estimation of Linear Models

Steven Portnoy
Roger Koenker

COMMERCE LIBRARY
SEP 11 1987
UNIVERSITY OF ILLINOIS
URBANA-CHAMPAIGN

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

BEBR

FACULTY WORKING PAPER NO. 1372

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

July 1987

Adaptive L-Estimation of Linear Models

Steven Portnoy
University of Illinois

Roger Koenker, Professor
Department of Economics

Adaptive L-Estimation of Linear Models[1]

Steven Portnoy

Roger Koenker

Abstract

Asymptotically efficient (adaptive) estimators for the slope parameters of the linear regression model are constructed based upon the "regression quantile" statistics suggested by Koenker and Bassett (1978). The estimators are natural analogues of the adaptive L-estimators of location of Sacks (1974), but employ kernel-density type estimators of the optimal L-estimator weight function.

Version: July 13, 1987.

Keywords: Regression quantiles, kernel density estimation, adaptive estimation, linear models.

1. Research supported by NSF Grants SES-840857 and MCS 8503785.

1. *Introduction*

The existence of asymptotically efficient estimators of a Euclidean parameter, β, in the presence of an infinite-dimensional nuisance parameter, F, has attracted considerable recent attention. The problem, formulated by Stein (1956) of asymptotically estimating β as well as when F is unknown as when it is known, has been treated in increasing generality. In a remarkable confluence of papers Beran (1974), Sacks (1974), and Stone (1975) independently proposed *adaptive* R-, L-, and M-estimators, respectively, of the center of symmetry of an unknown (symmetric) distribution. In his 1980 Wald lecture, Bickel (1982), developing the approach of Stein (1956) extended adaptation to a broad array of problems. In particular, he proposed an adaptive M-estimator for the parameters of the linear model

$$y_i = x_i'\beta_0 + e_i \tag{1.1}$$

with $\{x_j' = (x_{j1}, ..., x_{jp})\}$ a sequence of known p-vectors, $\beta_0 \in \mathbf{R}^p$ an unknown regression parameter to be estimated, and $\{u_i\}$ a sequence of independent random variables with common distribution function F. When F is symmetric, Bickel constructed an adaptive estimator of the entire vector β_0. Dropping the symmetry condition, he further showed that if the design contains an intercept, that is, $x_i' = (1, \dot{x}_i')$ so,

$$y_i = x_i'\beta_0 + e_i = \alpha + \dot{x}_i'\gamma + e_i \tag{1.2}$$

then the $(p-1)$-vector of "slope" parameters can be adaptively estimated. Manski (1984) reviewed these results, and offers some extensions to non-linear regression models. Newey (1987) has recently proposed adaptive method-of-moment type estimators for the linear model which are asymptotically efficient under rather weak regularity conditions. Hogg (1980) has also proposed various partially adaptive methods based on M-estimators, and de Jongh and de Wet (1986) have recently suggested an adaptive choice of the trimming proportion for trimmed least squares estimators.

In this paper we propose fully adaptive L-estimators for the slope parameters of the linear model, under the least restrictive assumptions possible on F (needed only to make the asymptotic efficiency well defined). These results extend results of Sacks (1974) to the case of linear regression and Koenker and Portnoy (1987) to the adaptive case. In the remainder of this section, we introduce notation and state our main results. Section 2 gives a detailed treatment of our construction of the adaptive estimator. Section 3 treats the problem of constructing a satisfactory estimate of the score function. Section 4 constructs a practical version of an adaptive L-estimator and describes a small monte-carlo experiment designed to evaluate the performance of the adaptive L-estimator in moderate-sized samples. We conclude that a practical adaptive L-estimator can be constructed for the slope parameters of the linear model. The estimator achieves high finite-sample efficiency in a wide variety of error situations and outperforms standard robust methods in all situations we investigated. Substantial gains in efficiency are achieved relative to simpler robust procedures in asymmetric error situations.

Let X_n denote the $(n \times p)$-matrix with i^{th} row x_i'; we will assume throughout that $n^{-1}X_n'X_n \to Q$, a positive definite matrix. The Euclidean norm of x will be denoted $\|x\|$ and $\lambda_1(M)$ will denote the largest eigenvalue for the matrix M. We will focus attention on Bickel's (1982) example 3: the linear model (1.2) *with* an explicit intercept and *without* any symmetry condition on F. We also assume that the means have been subtracted in X_n so that $\sum \dot{x}_i = 0$ where $\dot{x}_i$ is the last $(p-1)$ coordinates of x_i. Thus, if Q is partitioned so that $\dot{Q}$ is the lower $(p-1) \times (p-1)$ corner, $\dot{Q}^{-1}$ is the corresponding corner of Q^{-1}. The following regularity condition on the sequence of designs $\{X_n\}$ will be maintained.

Condition X: There exist positive constants b, $\bar{b}$, $\underline{b}$, and c, such that

(1.) $\lambda_1(Q - n^{-1}X_n'X_n) \leq bn^{-1/4}$

(2.) $\sum_{i=1}^{n} \|x_i\|^3 \leq bn$

(3.) $\max_i \|x_i\| \leq bn^{1/4}$

(4.) $\inf_{\|\delta\|=1} \#\{i: \underline{b} \leq x_i'\delta \leq \bar{b}\} \geq cn$

In Portnoy (1984) it is shown that such conditions are satisfied for a broad class of random designs, as well as for ANOVA designs when the number of observations per cell tends to infinity. On F we require only:

Condition F: F is absolutely continuous with finite, non-zero Fisher information $I(F)$.

Our methods are based on the regression quantiles of Koenker and Bassett (1978) which solve for $t \in [0,1]$

$$\min_{\beta \in R^p} \sum_{i=1}^{n} \rho_t(y_i - x_i'\beta). \tag{1.3}$$

where $\rho_t(u) = u\big(t - I(u < 0)\big)$. Let $\{\hat{\beta}_n(t) = (\hat{\alpha}_n(t), \hat{\gamma}_n(t))\}$ denote the sequence of regression quantile processes so defined. In the Appendix, a uniform Bahadur representation with explicit remainder is established for $\hat{\beta}_n(t)$. This result strengthens somewhat similar results of Jurečková and Sen (1984) and Koenker and Portnoy (1987).

Our adaptive estimator, T_n, of γ is a linear function of $\hat{\beta}_n(t)$, that is, we consider

$$T_n = \int_0^1 \hat{\gamma}_n(t)\, \hat{J}_n(t)\, dt \tag{1.4}$$

where $\hat{J}_n(t)$ is an estimate of the *optimal* score function

$$J_0(t) = \psi'(F^{-1}(t))$$

where $\psi(x) = -L'(x)$ and $L(x) = \ln f(x)$. Theorem 2.1 provides conditions on $\hat{J}_n(t)$ which make T_n adaptive for any F satisfying Condition F. A kernel estimator $\hat{J}_n(t)$ is constructed in Section 3 which satisfies the condition of Theorem 2.1, verifying our claim. Some further remarks on practical aspects of estimating $J_0(t)$ are contained in Section 4.

Our estimator of the optimal score function is based on the estimators of the conditional quantile and conditional distribution functions introduced in Bassett and Koenker (1982). Denoting the set of solutions to (1.3) by $\hat{B}_n(t)$, we may define a natural estimator of the t^{th} conditional quantile of Y given x, as,

$$\hat{Q}_n(t \mid x) = \inf \{x'b \mid b \in \hat{B}_n(t)\}. \tag{1.5}$$

And correspondingly,

$$\hat{F}_n(y \mid x) = \sup \{t \in [0,1] \mid \hat{Q}(t \mid x) \leq y\}, \tag{1.6}$$

affords a natural estimator of the conditional distribution function. At the mean of the design, $\bar{x} = n^{-1}\Sigma x_i$, $\hat{Q}(u \mid \bar{x})$ is a proper quantile function (a non-decreasing, left-continuous, step function on $u \in [0,1]$ (see Bassett and Koenker (1982) Theorem 2.1), so $\hat{F}_n(y) \equiv \hat{F}_n(y \mid \bar{x})$ is a proper (non-decreasing, right-continuous step-function on $y \in \mathbf{R}$) distribution function. $\hat{F}_n$ behaves asymptotically exactly like a sample distribution function (see Portnoy (1984)). The results of Section 3 give methods of estimating $J_0(t)$, based on $\hat{F}_n(y)$ which satisfy the conditions for adaptation of T_n given in Section 2.

2. *The Adaptive Estimators*

In order to treat asymptotics for L-estimators it is necessary to have smooth, positive densities. Following Stone (1975) this may be accomplished in great generality by convolving the original error distribution with a vanishingly small smooth contaminant. In particular, define

$$\tilde{u}_i = u_i + \frac{W_i}{s} + \frac{W_i'}{t}, \qquad Y_i = x_i'\beta + \tilde{u}_i \tag{2.1}$$

where $\{W_i\}$ and $\{W_i'\}$ are independent i.i.d. sequences (independent of u_i) with density

$$g(w) = \frac{c}{(1 + \rho(w))^2} \qquad -\infty < w < \infty. \tag{2.2}$$

Here $\rho(w)$ is an even continuously three times differentiable, positive function, increasing on

[0, 1], with $\rho(w) = |w|$ for $|w| \geq 1$. Let $G(w)$ denote the c.d.f. corresponding to g, and define (for $F \in \mathbf{F}$)

$$\begin{aligned} f_t(x) &= t \int g(t(x-y))\, dF(y) \\ f_s(x) &= s \int g(s(x-y))\, dF_t(y) \\ F_t(x) &= \int G(t(x-y))\, dF(y) \\ F_s(x) &= \int G(s(x-y))\, dF_t(y). \end{aligned} \tag{2.3}$$

That is, f_s and F_s are the density and c.d.f. for $\tilde{u}_i$; and f_t and F_t are the density and c.d.f. for $u_i + W_i/t$. Lastly define for fixed $\eta \leq \frac{1}{2}$ and arbitrary $b > 0$,

$$s_n = (\log n)^{\eta}, \qquad t_n = (\log n)^{b} \tag{2.4}$$

Note that the subscript n on s_n and t_n will often be suppressed.

Furthermore, since the uniform Bahadur representation (Theorem A.1) holds only on a compact subinterval of [0, 1], the interval of integration must also be restricted to a subinterval. Thus, for fixed $\delta \geq 0$, $0 \leq \epsilon \leq \delta + \eta < \frac{1}{2}$, and $a < \frac{1}{2}$, define (for $F \in \mathbf{F}$)

$$\begin{aligned} \alpha_n &= (\log n)^{-\epsilon} + F_t(-\tfrac{1}{2}(\log n)^{\delta}) + 1 - F_t(+\tfrac{1}{2}(\log n)^{\delta}) \\ \hat{\alpha}_n &= n^{-a} + (\log n)^{-\epsilon} + \hat{F}_n^{*}(-\tfrac{1}{2}(\log n)^{\delta}) + 1 - \hat{F}_n^{*}(+\tfrac{1}{2}(\log n)^{\delta}), \end{aligned} \tag{2.5}$$

where $\hat{F}_n^{*}$ is the Koenker-Bassett c.d.f. estimator (see (1.6)) based on observations $Y_i + W_i'/t_n$. Also let $\hat{F}_n$ denote the Koenker-Bassett c.d.f. estimator based on observations $\tilde{Y}_i = Y_i + W_i/s_n + W_i'/t_n$ (that is, $\hat{F}_n$ estimates F_s). Now define the adaptive (slope parameter) estimator:

$$T_n = \frac{\int_{\hat{\alpha}_n}^{1-\hat{\alpha}_n} \hat{\gamma}_n(t)\, \hat{J}_n(t)\, dt}{\int_{\hat{\alpha}_n}^{1-\hat{\alpha}_n} \hat{J}_n(t)\, dt} \tag{2.6}$$

where $\hat{J}_n(t)$ is any appropriately consistent estimator of the score function $J_{s_n}(t) \equiv L_{s_n}'(F_{s_n}^{-1}(t))$. An appropriate example (satisfying (2.7)) generated by kernel estimation based

on $\hat{F}_n$ is given in section 3.

Theorem 2.1 Let $\hat{J}_n(t)$ *be an estimator of* $J_s(t)$ *satisfying*

$$\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t) - J_s(t)|\, dt = o_p((\log n)^{-(2\delta+\eta)}) \tag{2.7}$$

Then for any $F \in \mathrm{F}$

$$\sqrt{n}\,(T_n - \gamma) \xrightarrow{D} N_{p-1}(0, \dot{Q}^{-1}/\mathrm{I}(F))$$

where I *is the Fisher information for* F.

This theorem will be proved after some preliminary properties of f_s are developed. The following Lemmas each assumes the hypotheses of Theorem 2.1 and that $F \in \mathrm{F}$.

Lemma 2.1 Given α_n *defined by* (2.5), *define*

$$x_n = \max\{-F_s^{-1}(\alpha_n), F_s^{-1}(1 - \alpha_n)\}. \tag{2.8}$$

Then there is a constant c^* *such that for* $B_n \le (\log n)^\delta$,

$$\inf\{f_s(x): -x_n - B_n \le x \le x_n + B_n\} \ge c^*(\log n)^{-(2\delta+\eta)}.$$

Proof. Note that (by (2.2)),

$$\begin{aligned} F_s(-x_n) &= \int G(s(x_n - y))\, dF_t(y) \le G(-\tfrac{1}{2}x_n) + P\{|u + W/t| \ge \tfrac{1}{2}x_n\} \\ &\le \frac{c^*}{(1 + \tfrac{1}{2}sx_n)} + F_t(-\tfrac{1}{2}x_n) + 1 - F_t(\tfrac{1}{2}x_n) \end{aligned} \tag{2.9}$$

and a similar inequality holds for $1 - F_s(x_n)$. Hence, from (2.8) $\alpha_n = F_s(-x_n)$ or $\alpha_n = 1 - F_s(x_n)$; and, if x_n were larger than $(\log n)^\delta$, (2.9) would be contradicted by (2.5) (for n large enough). Thus, it follows that (for n large enough),

$$0 \le x_n \le (\log n)^\delta \quad \text{and} \quad x_n \to +\infty \tag{2.10}$$

(since $\alpha_n \to 0$ by (2.5))/ Now (for $x > 0$)

$$f_s(x) = \int \frac{cs}{(1 + \rho(s(x - y)))^2}\, dF_t(y) \ge \frac{cs}{(1 + \rho(sx))^2} P\{u + W/t \le 0\},$$

and, hence, for $|x| \le 2(\log n)^{\delta}$, with $c^* = P\{u + W/t \le 0\}$,

$$f_s(x) \ge \frac{c^* s}{(1 + \rho(2s(\log n)^{\delta}))^2} = \frac{c^* s}{(1 + 2s(\log n)^{\delta})^2}$$

and the result follows from (2.4). ■

Lemma 2.2. For constants c_ν $(\nu = 0, 1, 2, 3)$ *with* $c_0 = c$ *in* (2.2),

$$|f_s^{(\nu)}(x)| \le c_\nu s^{\nu+1} \quad \textit{and} \quad |f_t^{(\nu)}(x)| \le c_\nu s^{\nu+1}$$

uniformly in x.

Proof. Differentiate $f_s(x)$ or $f_t(x)$ (see (2.3)) under the integral and use the fact that derivatives of ρ are uniformly bounded. ■

Lemma 2.3. $f_{s_n}(x_n) \to 0$ *and* $f_{s_n}'(x_n) \to 0$ as $n \to \infty$.

Proof. As in (2.9) (using (2.4) and (2.10)),

$$|f_s(x_n)| \le \frac{cs}{(1 + \tfrac{1}{2}sx_n)^2} + \bar{c}\,\{F_t(-\tfrac{1}{2}x_n) + 1 - F_t(\tfrac{1}{2}x_n)\} \to 0$$

$$|f_s'(x_n)| \le \frac{c^* s^2}{|1 + \tfrac{1}{2}sx_n|^3} + \bar{c}^*\{F_t(-\tfrac{1}{2}x_n) + 1 - F_t(\tfrac{1}{2}x_n)\} \to 0. \quad ■$$

Lemma 2.4. $\int_{\alpha_n}^{1-\alpha_n} J_s(t)\,dt \to \mathrm{I}(F)$ *as* $n \to \infty$.

Proof. A slight modification of the proof of Theorem 4.1 in Stone (1975) provides the result here. ■

Lemma 2.5. As $n \to \infty$,

$$\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t)|\,dt = \mathbf{O}_p(s_n^2) = \mathbf{O}_p((\log n)^{2\eta})$$

Proof. By condition (2.7), we need only consider

$$\int_{\alpha_n}^{1-\alpha_n} |J_s(t)|\,dt \le \int_{-x_n}^{x_n} |L_s''(x)|\, f_s(x)\,dx$$
$$\le \int_{-x_n}^{x_n} |f_s''(x)|\,dx + \int_{-x_n}^{x_n} (L_s'(x))^2 f_s(x)\,dx \tag{2.11}$$

Now differentiating f_s in (2.3) twice (under the integral) and using the fact that derivatives of ρ are bounded,

$$|f_s''(x)| \le \int \frac{2cs^3|\rho''(s(x-y))|}{(1+\rho(s(x-y)))^3}\,dF_t(y) + \int \frac{3cs^3(\rho'(s(x-y)))^2}{1+\rho(s(x-y)))^4}\,dF_t(y)$$
$$\le c_1 s^2 \int \frac{cs}{(1+\rho(s(x-y)))^2}\,dF_t(y) = c_1 s^2 f_s(x).$$

Hence, the first term in (2.11) is $\mathbf{O}(s^2)$. The last term in (2.11) converges to $\mathbf{I}(F)$ by Lemma 2.4; and, thus, the desired result follows. ■

Lemma 2.6 Let α_n *and* $\hat{\alpha}_n$ *be given by* (2.5) *and assume* $F \in \mathbf{F}$. *Then, with probability tending to one (with* $a < \frac{1}{2}$ *as defined in (2.5)),*

$$\alpha_n \le \hat{\alpha}_n \le \alpha_n + 2n^{-a}.$$

Proof. By proposition 3.1, $|\hat{F}_n^*(x) - F_t(x)| = \mathbf{O}_p(n^{-1/2})$ uniformly for $F_s^{-1}(\alpha_n) \le x \le F_s^{-1}(1-\alpha_n)$. By equations (2.10) and (2.8), $\pm \frac{1}{2}(\log n)^b$ lies in this interval, and the result follows immediately. ■

Lemma 2.7. As $n \to \infty$, $\int_{S_n} |J_s(t)|\,dt = \mathbf{O}(n^{-a/2})$, *where* $S_n = (\alpha_n, \hat{\alpha}_n) \bigcup (1-\hat{\alpha}_n, 1-\alpha_n)$.

Proof. Following the argument of Lemma 2.5 and using Lemma 2.6, with probability tending to one

$$\int_{\alpha_n}^{\hat{\alpha}_n} |J_s(t)|\,dt \le c_1 s^2 (F_s^{-1}(\alpha_n + 2n^{-a}) - F_s^{-1}(\alpha_n)) + \int_{-x_n}^{F_s^{-1}(\alpha_n+2n^{-a})} (L_s'(x))^2 f_s(x)\,dx.$$

From Lemma 2.1 (and the mean value theorem) the first term is of order $\mathbf{O}(\log n)^b\, n^{-a} = \mathbf{O}(n^{-a/2})$. A similar argument shows that the second term has this same order. The same argument also applies to the integral from $(1-\hat{\alpha}_n)$ to $(1-\alpha_n)$. ■

Proof of Theorem 2.1. From (2.6)

$$\sqrt{n}\,(T_n - \gamma) = \frac{\int_{\hat{\alpha}_n}^{1-\hat{\alpha}_n} \sqrt{n}\,(\hat{\gamma}_n(t) - \gamma)\,\hat{J}_n(t)\,dt}{\int_{\hat{\alpha}_n}^{1-\hat{\alpha}_n} \hat{J}_n(t)\,dt} \equiv \frac{A_n}{B_n} \tag{2.12}$$

By Lemmas 2.4 and 2.7 and condition (2.7), the denominator, B_n, tends to $\mathbf{I}(F)$ in probability; so it remains to consider the numerator. Define

$$U_n = \frac{1}{\sqrt{n}}\,\dot{Q}^{-1} \sum_{i=1}^{n} \dot{x}_i K_{in}(t), \quad K_{in}(t) = t - I(\tilde{u}_i \le F_\theta^{-1}(t)) \tag{2.13}$$

Then, by Theorem A.1,

$$\sqrt{n}\,|\hat{\gamma}_n(t) - \gamma - \frac{1}{\sqrt{n}} U_n(t)(f_\theta(F_\theta^{-1}(t)))^{-1}| \le (n^{-1/4}(\log n)B(X, F_\theta) + n^{-1/2}b_1(X))/f_\theta(F_\theta^{-1}(t))$$

on $(\alpha_n, 1 - \alpha_n)$ except with probability bounded by $q(X, F)$ (see Lemma (A.3)). By Lemmas 2.1 and 2.2, uniformly on $(\alpha_n, 1 - \alpha_n)$,

$$\frac{B(X,F_\theta)}{f_\theta(F_\theta^{-1}(t))} = \mathbf{O}((\log n)^{\eta + 3(2\delta + \eta)}), \qquad q(X, F) = \mathbf{O}\left[\frac{1}{\sqrt{n}}\, e^{c^2 b_4}\,(\log n)^{2(\delta+\eta)}\right] \to 0.$$

Therefore (using Theorem A.1), with probability tending to one, the numerator in (2.12) satisfies (since $\alpha_n \le \hat{\alpha}_n$ in probability)

$$|A_n - \int_{\alpha_n}^{1-\alpha_n} U_n(t) J_s(t)/f_s(F_s^{-1}(t))dt|$$

$$\leq \mathbf{O}_p(n^{-1/4}(\log n)) \frac{\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t)|\, dt}{\inf_{\alpha_n} f_s(x)} + \frac{\sup_t |U_n(t)|}{\inf_{\alpha_n} f_s(t)} \int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t) - J_s(t)|\, dt \tag{2.14}$$

$$+ \frac{\sup_t |U_n(t)|}{\inf_{\alpha_n} f_s(t)} \int_{S_n} |J_s(t)|\, dt$$

where $S_n = (\alpha_n, \hat{\alpha}_n) \bigcup (1 - \hat{\alpha}_n, 1 - \alpha_n)$. By condition (2.7) and Lemma 2.5, $\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t)|\, dt = O_p(s_n^2)$; and, hence, by (2.4) and Lemma 2.1, the first term in (2.14) tends to zero in probability. Using an invariance principle for $U_n(t)$ (e.g., see Koul (1969), Theorem A.3), $\sup_t |U_n(t)| = \mathbf{O}_p(1)$. Thus, combining Lemma 2.1 and condition (2.7), the second term in (2.14) also tends to zero in probability. Lastly, the third term converges to zero by Lemmas 2.1 and 2.7. Therefore, the right side of (2.14) tends to zero in probability; and it remains to consider

$$V_n \equiv \int_{\alpha_n}^{1-\alpha_n} U_n(t)\, J_s(t)/f_s(F_s^{-1}(t))\, dt.$$

Fix $t \in R^p$ and consider $t'V_n$. define $a_{in} = t'\dot{Q}^{-1}\dot{x}_i/\sqrt{n}$. Then

$$\sum_{i=1}^{n} a_{in}^2 \to t'\dot{Q}^{-1}t \quad \text{as} \quad n \to \infty. \tag{2.15}$$

and $t'V_n$ is a weighed sum of n i.i.d. random variables (see (2.13)):

$$t'V_n = \sum_{i=1}^{n} a_{in} \int_{\alpha_n}^{1-\alpha_n} K_{in}(t)\, J_s(t) / f_s(F_s^{-1}(t))\, dt$$

To apply the Liapounov Central Limit Theorem, compute third moments: since $|K_{in}(t)| \leq 2$

$$E\,|t'V_n|^3 \le 8\sum_{i=1}^{n}|a_{in}|^3\left\{\int_{\alpha_n}^{1-\alpha_n}|J_s(t)|/f_s(F_s^{-1}(t))\,dt\right\}^3 \le \sum_{i=1}^{n}|a_{in}|^3\,\mathbf{O}((\log n)^{2\delta+3\eta})$$

where Lemma 2.1 and Lemma 2.5 are applied. Lastly, from condition X3, the definition of a_{in}, and (2.15),

$$E\,|t'V_n|^3 \le \sum a_{in}^2\mathbf{O}(n^{-1/4}(\log n)^b) \to 0 \quad \text{as} \quad n \to \infty.$$

So by Liapounov's theorem (e.g., see Breiman (1968), p. 275), it remains to check that the variance converges to $t'\dot{Q}^{-1}t \cdot \mathbf{I}(F)$. Direct calculation gives

$$\sigma_n \equiv \frac{\operatorname{Var} t'V_n}{t'\dot{Q}^{-1}t} = \int_{\alpha_n}^{1-\alpha_n}\int_{\alpha_n}^{1-\alpha_n}\frac{\min(t,t') - tt'}{f_s(F_s^{-1}(t))f_s(F_s^{-1}(t'))}J_s(t)J_s(t')\,dt\,dt'$$

$$= \int_{x_n}^{y_n}\int_{x_n}^{y_n}\{\min(F_s(x),F_s(y)) - F_s(x)F_s(y)\}L_s''(x)L_s''(y)\,dx\,dy$$

(where $x_n = F_s^{-1}(\alpha_n)$, $y_n = F_s^{-1}(1-\alpha_n)$). Let σ_s denote the above double integral with $x_n = -\infty$ and $y_n = \infty$. Then $\sigma_s - \sigma_n$ can be expressed as the sum of integrals over rectangles disjoint from $(x_n, y_n) \times (x_n, y_n)$. Consider one such integral: the integral over $(-\infty, x_n) \times (x_n, \infty)$. Integrating by parts,

$$\left|\int_{-\infty}^{x_n}\int_{x_n}^{\infty}F_s(y)\,(1-F_s(x))\,L_s''(x)\,L_s''(y)\,dx\,dy\right|$$

$$= |\{-(1-F_s(x_n))L_s'(x_n) - f_s(x_n)\}\{F_s(x_n)L_s'(x_n) - f_s(x_n)\}|$$

$$\le (L_s'(x_n))^2F_s(x_n) + |L_s'(x_n)|f_s(x_n) + f_s^2(x_n).$$

By Lemma 2.3, $f_s^2(x_n) \to 0$ and $|L_s'(x_n)|f_s(x_n) = |f_s'(x_n)| \to 0$ as $n \to \infty$. Also by L'Hospital's rule,

$$\lim F_\theta(x_n)(L_\theta'(x_n))^2 = \lim F_\theta(x_n)L_\theta'(x_n) \lim \frac{f_\theta'(x_n)}{f_\theta(x_n)}$$

$$= \lim F_\theta(x_n)L_\theta'(x_n) \cdot \lim \frac{f_\theta(x_n)}{F_\theta(x_n)}$$

$$= \lim f_\theta'(x_n) = 0,$$

by Lemma 2.3. Treating other contributions to $|\sigma_n - \sigma_\theta|$ similarly, we see that $|\sigma_n - \sigma_\theta| \to 0$ as $n \to \infty$. But integrating by parts, $\sigma_\theta = \int_{-\infty}^{\infty} L_\theta''(x)\, f_\theta(x)\, dx \to \mathrm{I}(F)$ as $n \to \infty$ (by Lemma 2.4). Therefore, $\sigma_n \to \mathrm{I}(F)$; and, hence, $V_n \xrightarrow{D} \mathrm{N}_{p-1}(0, \mathrm{I}(F)\, \dot{Q}^{-1})$. As noted above, this implies A_n (in (2.12)) has the same limiting distribution. Therefore $A_n/B_n \xrightarrow{D} \mathrm{N}_{p-1}(0, \dot{Q}^{-1}/\mathrm{I}(F))$, and the proof is complete. ■.

3. *An Appropriate Estimator of the Score Function*

Here, as in section 2; we assume that the errors are distributed according to F_θ defined in (2.3) with $F \in \mathrm{F}$. For such smooth F_θ; it is relatively easy to construct an estimator, $\hat{J}_n(t)$ of $J_\theta(t)$ satisfying (2.7) by using appropriate density estimators based on $\hat{F}_n$. Since (2.7) requires only logarithmic convergence, the following conditions on the density estimators will be seen to be sufficient. Let $s_n = (\log n)^\eta$ (as in (2.4)),

$$U_n = \{x: F_\theta^{-1}(\alpha_n) - B \le x \le F_\theta^{-1}(1 - \alpha_n) + B\} \tag{3.1}$$

for any constant B, and define $K_n = (\log n)^{-(2\delta+\eta)}$ (so that, by Lemma 2.1, $\{\inf_{U_n} f_\theta(x)\}^{-1} = \mathrm{O}(1/K_n)$). Suppose there are density estimators, $\hat{f}_n(x)$ (with derivatives $\hat{f}_n^{(\nu)}(x)$), and (smooth) c.d.f. estimators, $\tilde{F}_n(x)$ (generally the integral of $\hat{f}_n$) such that for $\nu = 0$, 1, 2, 3,

$$\sup_{U_n} |\hat{f}_n^{(\nu)}(k) - f_\theta^{(\nu)}(x)| = \mathrm{o}_p((s_n/K_n)^{-6}) = \mathrm{o}_p((\log n)^{-10(\delta+\eta)}) \tag{3.2}$$

$$\sup_{U_n} |\tilde{F}_\theta^{-1}(F_\theta(x)) - x| = \mathrm{o}_p((s_n/K_n)^{-4}) = \mathrm{o}_p((\log n)^{-8(\delta+\eta)}) \tag{3.3}$$

(where f_s and F_s are given by (2.3)). As in section 2, define $L_s(x) = \log f_s(x)$, $\hat{L}_n(x) = \log \hat{f}_n(x)$, $J_s(t) = L_s''(F_s^{-1}(t))$, and $\hat{J}_n(t) = \hat{L}_n''(\hat{F}_n^{-1}(t))$.

Lemma 3.1. If (3.2) *holds, then*

$$\sup_{U_n} |\hat{L}_n''(x) - L_s''(x)| = o_p((\log n)^{-(2\delta+\eta)}).$$

Proof. First note that by (3.2) and Lemma 2.1,

$$\inf_{U_n} \hat{f}_n(x) \geq c^* K_n - o_p(1). \tag{3.4}$$

Hence, $\{\inf_{U_n} \hat{f}_n(x)\}^{-1} = O_p(1/K_n)$. Similarly, by (3.2) and Lemma 2.2, we also have

$$|\hat{f}_n^{(\nu)}(x)| \leq c_\nu S^{\nu+1} \quad \text{for} \quad x \in U_n. \tag{3.5}$$

Therefore, letting Δf denote absolute differences between $\hat{f}_n$ and f_s (and their derivatives), and (with n suppressed) writing $L''(x) = f''(x)/f(x) - (f'(x)/f(x))^2$,

$$\begin{aligned}
\sup_{U_n} |\hat{L}_n''(x) - L_s''(x)| &\leq O\left(\frac{\Delta f''}{K} + \frac{\sup f'' \Delta f}{K^2} + \frac{\sup(\hat{f}' + f')\Delta f'}{K^2} \right. \\
&\qquad \left. + \frac{\sup f'^2 \sup(\hat{f} + f)\Delta f}{K^4} \right) \\
&= O_p\left(\frac{S_n^5}{K_n^4} \right) \cdot o_p\left(\left(\frac{S_n}{K_n} \right)^{-5} \right) \\
&= o_p(K_n). \quad \blacksquare
\end{aligned}$$

Theorem 3.1. If (3.2) *and* (3.3) *hold, then*

$$\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t) - J_s(t)| \, dt = o_p(\log n^{-(2\delta+\eta)}).$$

Proof. Changing variables using $t = F_s^{-1}(x)$ and letting $x_n = F_s^{-1}(\alpha_n)$, $y_n = F_s^{-1}(1 - \alpha_n)$,

$$\int_{\alpha_n}^{1-\alpha_n} |\hat{J}_n(t) - J_s(t)|\, dt = \int_{x_n}^{y_n} |\hat{L}_n''(\tilde{F}_n^{-1}(F_s(x))) - L_s''(x)|\, f_s(x)\, dx$$

$$\leq \int_{x_n}^{y_n} |\hat{L}_n''(x) - L_s''(x)|\, f_s(x)\, dx + \int_{x_n}^{y_n} \left| \int_{x}^{\tilde{F}^{-1}(F_s(x))} \hat{L}'''(u)\, du \right| f_s(x)\, dx.$$

The first term has the desired order by Lemma 3.1. By Lemmas 2.1 and 2.2 and (3.4) and (3.5),

$$\sup_{U_n} |\hat{L}'''(x)| = \mathbf{O}_p(s_n^4/K_n^3) \tag{3.6}$$

Hence, by condition (3.3) and (3.6), the inner integral in the second term above is $\mathbf{O}_p(s_n^4/K_n^3) \cdot \mathbf{o}_p((s_n/K_n)^{-4}) = \mathbf{o}_p(K_n)$; and the result follows. ■

Lastly, estimates $\hat{f}_n$ satisfying (3.2) and (3.3) need to be constructed. In fact, it is generally easy to construct estimates where the error terms are even smaller than those required in conditions (3.2) and (3.3). For example, if there is a c.d.f. estimator, $\hat{F}_n(x)$, satisfying

$$\sup_{U_n} |\hat{F}_n(x) - F_s(x)| = \mathbf{O}_p(n^{-a}) \quad \text{for some} \quad d > 0 \tag{3.7}$$

then kernel estimators satisfying (3.2) and (3.3) can be constructed (and similarly for estimating F_t). We first show that (3.7) holds for $a = \frac{1}{2}$ for the Koenker-Bassett c.d.f. estimator, $\hat{F}_n$, given by (1.6) based on observations $\tilde{Y}_i$. However, it is no harder to show that the empirical distribution of residuals from any estimator, $\hat{\beta}$ (with $\hat{\beta}$ consistent at rate n^{-a}) will also satisfy (3.7).

Proposition 3.1: *Assume that the result of Theorem A.1 holds. Then condition* (3.7) *holds for* $F \in \mathbf{F}$ *with* $a = \frac{1}{2}$.

Proof. By Theorem A.1 and Lemma 2.1 and 2.2,

$$\sup_{U_n} |\hat{F}_n(x) - F_s(x)| \leq \sup_{U_n} \left| \frac{1}{n} \sum_{i=1}^{n} I(\tilde{u}_i \leq x) - F_s(x) \right| + \mathbf{O}_p(n^{-3/4} (\log n)^b)$$

for some $b > 0$. By Kolmogorov's result (e.g., see Breiman (1968), p. 287) the sup on the right is $\mathbf{O}_p(n^{-1/2})$; and, hence, (3.7) holds. The same argument works for $|\hat{F}_n^*(x) - F_t(x)|$ where $\hat{F}_n^*$

is based on $Y_i + W_i/t$. ■

Now, let $k(x)$ be a kernel which is a (symmetric) density with support in [-1, 1] such that $|k^{(\nu)}(x)| \leq b$ (for some $b > 0$) uniformly for all x and $\nu = 0, 1, 2, 3, 4$. Given $\hat{F}_n(x)$ satisfying (3.7) define

$$\begin{aligned} \hat{f}_n(x) &= r_n \int_{-\infty}^{\infty} k(r_n(x - y)\, d\hat{F}_n(y) \\ \bar{F}_n(x) &= \int_{-\infty}^{x} \hat{f}_n(x)\, dx, \end{aligned} \tag{3.8}$$

where

$$r_n = n^{a_0} \quad \text{with} \quad a_0 < a/4. \tag{3.9}$$

Lemma 3.2. If (3.7) *holds, then* (3.2) *holds for estimates given by* (3.8).

Proof. Integrating by parts, for $\nu = 0, 1, 2, 3$,

$$\hat{f}_n^{(\nu)}(x) = r^{\nu+2} \int_{-\infty}^{\infty} k^{(\nu+1)}(r(x - y))\, \hat{F}_n(y)\, dy.$$

Therefore,

$$\begin{aligned} |f_n^{(\nu)}(x) - f_s^{(\nu)}(x)| \leq\ & r^{\nu+2} \int_{-\infty}^{\infty} k^{(\nu+1)}(r(x - y))|\hat{F}_n(y) - F_s(y)|\, dy \\ & + \Big| \int_{-\infty}^{\infty} r^{\nu+2}\, k^{(\nu+1)}(r(x - y))\, F_s(y)\, dy - f_s^{(\nu)}(x) \Big|. \end{aligned} \tag{3.10}$$

By (3.7) and the conditions on k, the supremum of the first term above is of order $r^{\nu+1} \cdot \mathbf{O}_p(n^{-a})$, which decreases as a power of n by (3.9). For the second term, integrating by parts yields

$$\int_{-\infty}^{\infty} r^{\nu+2} k^{(\nu+1)}(r(x-y)) F_s(y)\, dy = \int_{-\infty}^{\infty} rk(r(x-y)) f_s^{(\nu)}(y)\, dy$$

$$= \int_{-\infty}^{\infty} k(u) f_s^{(\nu)}(x - \frac{u}{r})\, du$$

$$= f_s^{(\nu)}(x) - \frac{1}{r} \int_{-\infty}^{\infty} uk(u) f_s^{(\nu+1)}(\tilde{x}(u))\, du$$

Thus, by Lemma 2.2 and the conditions on k,

$$\sup_{U_n} \left| \int_{-\infty}^{\infty} r^{\nu+2} k^{(\nu+1)}(r(x-y)) F_s(y)\, dy - f_s^{(\nu)}(y) \right| = \mathbf{O}\left(\frac{s^{\nu+2}}{r} \right),$$

and, hence, the supremum of the second term in (3.10) also decreases as a power of n. Thus, (3.2) holds, in fact, with an error of order n^{-a^*} with $a^* < \min(a_0, a - 4a_0)$ (where a_0 defined in (3.9)). ∎

Lemma 3.3. *If* (3.7) *holds with* B *replaced by* 3B *in the definition of* U_n (3.1), *then* (3.3) *holds for estimates given by* (3.8).

Proof. Let $U_n(B)$ denote the set U_n in (3.1) with dependence on B explicit, and define

$$D_n = \sup_{U_n(3B)} |\hat{F}_n(x) - F_s(x)| = \mathbf{O}_p(n^{-a}).$$

Let $\epsilon > 0$ be given and choose n large enough so that by Lemma 2.1, (3.7), and (3.9),

$$\frac{1}{r} + D_n / \inf_{U_n(3B)} f_s(x) \le cn^{-a_1} \le B$$

for some $a_1 < a_0$ and constant c, with probability at least $1 - \epsilon$. Then since the support of k is contained in [-1, 1], (3.8) implies that for $y \in U_n(2B)$, with probability at least $1 - \epsilon$ (for n large enough),

$$\tilde{F}_n(y) \le \hat{F}_n(y + \frac{1}{r}) \le F_s(y + \frac{1}{r}) + D_n$$

$$\le F_s(y + \frac{1}{r} + D_n / \inf_{U_n(3B)} f_s(x)).$$

Now let $x = y + \frac{1}{r} + D_n / \inf_{U_n(3B)} f_s(x)$. Then for $x \in U_n(B)$,

$$x - \frac{1}{r} - D_n / \inf_{U_n(3B)} f_\theta(x) \leq \tilde{F}_n^{-1}(F_\theta(x))$$

or

$$x \leq \tilde{F}_n^{-1} F_\theta(x) + cn^{-a_1}$$

with probability at least $1 - \epsilon$ for n large enough. The reverse inequality follows similarly; and, hence, the result holds. ∎

4. Practical Experience

To assess the performance of adaptive L-estimation in practical applications, a small scale monte-carlo experiment was conducted. Before describing the experiment in detail, we should explicitly describe the version of the adaptive estiamtor (1.4) as it is employed in the experiment.

In Section 3, it is shown that the estimator $\hat{F}_n(y) \equiv \hat{F}_n(y \mid \bar{x})$ defined in (1.6) and described in detail in Bassett and Koenker (1982) and Portnoy (1984) satisfies the condition

$$\sup_{y \in U_n} |\hat{F}_n(y) - F_\theta(y)| = O_p(n^{-1/2}) \tag{4.1}$$

and $F_\theta(y)$ defined in Section 2, for U_n given in (3.1), and further, that kernel density estimators of f_θ and its derivatives based on $\hat{F}_n(y)$ can be used to achieve the sufficient condition (2.7) for an adaptive $\hat{J}_n(t)$ required by the estimator defined in (1.4).

Rather than randomly perturbing the observed y's as suggested by the theory of Sections 2 and 3, we have chosen instead to smooth $\hat{F}_n(y)$ directly by kernel methods. For appropriate choice of the kernel, this may be viewed as taking expectations with respect to the randomized estimator treated in Section 2, cf. Stone (1975). $\hat{F}_n(y)$ takes the form,

$$\hat{F}_n(y) = \sum_{i=1}^{n} p_i \, I(y \leq \xi_i) \tag{4.2}$$

for numbers $0 < p_1 < p_1 + p_2 < \sum_{i=1}^{m-1} p_i < 1$ and $\xi_1 < \xi_2 < \cdots < \xi_m$. So, we may write kernel

estimates of the density and its derivatives as

$$\hat{f}_n^{(\nu)}(x) = \sum_{i=1}^{m} p_i r_{in}^{\nu+1} k^{(\nu)}(r_{in}(x - \xi_i)) \tag{4.3}$$

where $k(\cdot)$ denotes a proper kernel and $r_{in}^{-1}: i = 1, \cdots, m$ are local bandwidth numbers which control the degree of smoothness of the estimate. The latter are chosen by the procedure outlined in Silverman (1986, pp. 101-2). A pilot estimate, $\tilde{f}(x)$, of the density is constructed based on a fixed bandwidth, say h. Then the local bandwidth factors,

$$\lambda_i = [\tilde{f}(\xi_i)/g]^{-\sigma}$$

are computed with $\log g = \sum p_i \log \tilde{f}(\eta_i)$. The sensitivity parameter, σ, controls the responsiveness of the local bandwidths

$$r_{in} = (h\lambda_i)^{-1}$$

to the pilot density. We have adopted the (standard) choice $\sigma = \frac{1}{2}$ after some brief experimentation with other values.

The choice of the kernel $k(\cdot)$ is critical to the success of the method. Guided by the theory of Section 2 we have chosen the Cauchy kernel,

$$k(x) = (\pi(1 + x^2))^{-1},$$

which has the salient characteristic that it tends to control the tail behavior of our estimated $J(\cdot)$ much more successfully than more conventional, thinner-tailed kernels.

Given the estimates (4.3), it is natural to define

$$\tilde{J}_n(t_i) = \left[\frac{\hat{f}_n^{(1)}(\xi_i)}{\hat{f}_n(\xi_i)}\right]^{1/2} - \left[\frac{\hat{f}_n^{(2)}(\xi_i)}{\hat{f}_n(\xi_i)}\right]^{1/2}, i = 1, 2, \cdots, m$$

where $t_i = \sum_{j=1}^{i} p_j$ is the cumulative mass associated with the quantile ξ_i. In theory *and practice* it is essential to trim the tails of the weight function so for a sequence $\alpha_n \to 0$, we compute,

$$\hat{J}_n(t_i) = \tilde{p}_i \tilde{J}_n(t_i) / \sum_{j=1}^{m} \tilde{p}_j \tilde{J}_n(t_j) \tag{4.4}$$

with $\tilde{p}_i = \max(\min(t_i, 1 - \alpha_n) - \max(t_{i-1}, \alpha_n), 0)$.

It remains only to describe the choice of (i) the initial window width, h; and (ii) the trimming proportion α. The latter is straightforward; we simply report results for *both* of the traditional trimming proportions $\alpha = 0.05$ and $\alpha = 0.1$. The theory of Section 3 suggests that $\alpha_n \to 0$ as a negative power of $\log n$; thus these traditional values should be reasonable for a wide range of sample sizes (say $n < 1000$). The choice of h is a delicate issue and warrants considerable further investigation. We began with a conventional rule for density estimation, see Silverman (1986, Section 3.4),

$$h = \kappa \min(s_1, s_2)/n^{1/5}$$

where s_1 and s_2 are alternative estimates of the dispersion of $\hat{F}_n(y)$: standard deviation, and (interquartile range)/1.34, respectively, and κ is a constant to be determined. The choice $\kappa = .9$ tuned to minimizing integrated mean-squared error of the normal density is clearly inappropriate in the present instance. Virtually imperceptible bulges in $\hat{f}$ give rise to violent oscillations in $\hat{J}$. We have adopted $\kappa = 2.5$ provisionally, although this tends to oversmooth to a significant degree in some cases. In Figures 4.1 and 4.2 we illustrate several estimated J functions for the Gaussian and Cauchy cases respectively for a bivariate linear model with 100 observations. The smooth curves in each case depict the "true" J.

We should emphasize at this point that many of the choices described above may be easily criticized. Indeed the choice of kernel estimation of J is itself questionable. Cox (1986) has proposed an elegant smoothing spline approach to the estimation of $f'(x)/f(x)$ which may prove attractive in the present instance as well, if a satisfactory approach can be found for controlling the tail behavior of the estimator. In some preliminary experiments we found this to be difficult. Clearly, many alternatives exist to the particular choice of initial and local bandwidths described above. We regard the current methods as simply illustrative

Figure 4.1
Three $\hat{J}$'s with Gaussian Errors

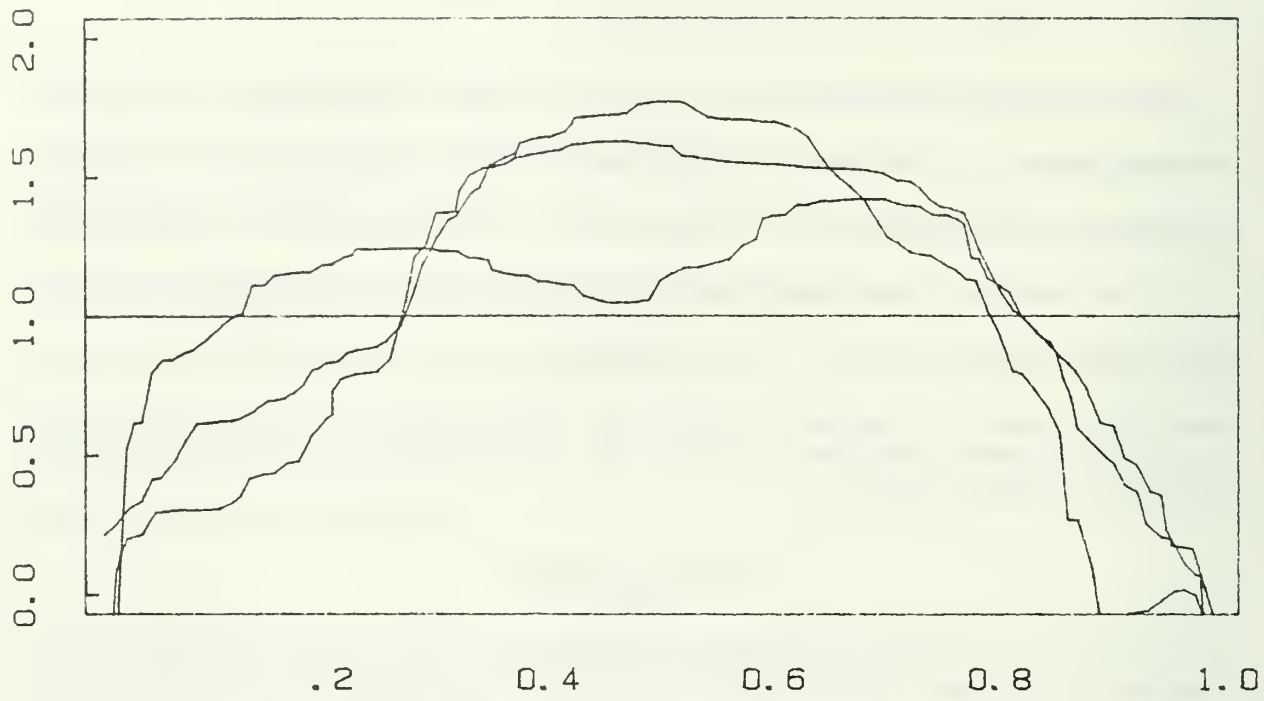
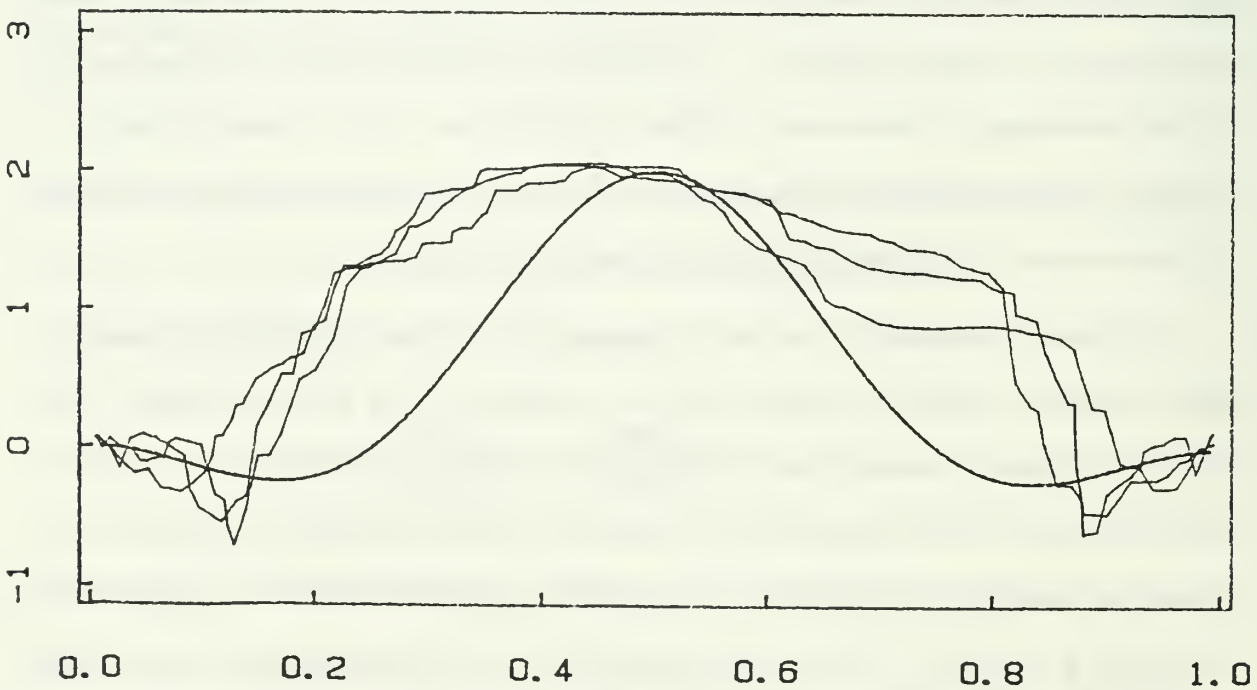


Figure 4.2
Three $\hat{J}$'s with Cauchy Errors



of one possible approach which yields quite promising results.

The experiment is (provisionally) limited to the bivariate linear model,

$$y_i = \alpha + \beta x_i + u_i$$

with the x_i drawn as i.i.d. Gaussian and u_i also i.i.d. from one of the distributions appearing in Table 4.1. Since asymmetric distributions are of substantial interest we restrict attention to the relative performance of several estimators of the *slope* parameter, β. To control computing costs we restrict attention to only a few competing estimators.

Once the regression quantile process, $\hat{\beta}(t)$, implicitly defined in (1.3) has been computed, it is easy to compute a variety of L-estimators. For example, the analogues of the trimmed means

$$\hat{\beta}_\alpha = (1 - 2\alpha)^{-1} \int_\alpha^{1-\alpha} \hat{\beta}(t)dt,$$

termed "trimmed regression quantiles" are readily calculated as in (4.4) setting $\tilde{J}_n(t) \equiv 1$ on $(\alpha, 1 - \alpha)$. These estimators are, asymptotically, closely related to the Huber M-estimators. We consider three members of this family: TRQ(.5), the l_1-estimator; TRQ(.25), a regression midmean; and TRQ(.1), the 10% trimmed regression quantile. In addition, for each case, we

Table 4.1

Distributions, Densities, and their Optimal J's

Name	Density[1]	Optimal J[2,3]
Gaussian	$\phi(x)$	1
Cauchy	$(\pi(1 + x^2))^{-1}$	$2(1 - Q^2(u))/(1 + Q^2(u))^2$
Uniform	$I_{[0,1]}(x)$	$.5\delta_0(u) + .5\delta_1(u)$
Laplace	$\frac{1}{2}e^{-\lvert x \rvert}$	$\delta_{1/2}(u)$
Exponential	$e^{-x} \quad x \geq 0$	$\delta_0(u)$
Lognormal	$x^{-1}\phi(\log x)$	$-\log(Q(u))/Q^2(u)$
Bimodal	$.5\phi(x - 3) + .5\phi(x + 3)$	$1 + 9\left[1 + \frac{(\phi(Q + 3) - \phi(Q - 3))^2}{(\phi(Q + 3) + \phi(Q - 3))^2}\right]$

1. $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$
2. $\delta_x(u)$ denotes the Dirac density with point mass 1 at x.
3. $Q = Q(u)$ denotes the quantile function corresponding to the density given in column 2.

compute the *optimal* L-estimator using the J function appearing in Table 4.1. Finally, we compute the ordinary least squares estimator (OLS) and the least maximum deviation (LMD), or l_∞ estimator. The latter is the maximum likelihood estimator for the uniform case, the sample midrange in the location model, and is readily computed by linear programming methods.

In Table 4.2 we report monte-carlo relative efficiencies for each of these estimators based on 1000 trials. The reported efficiencies are relative to the optimal L-estimator in each case defined by the J function given in Table 4.1. The random number generator was the portable version of the Marsaglia "superduper" generator as implemented in *S* (Becker and Chambers (1985)), so results should be reproducible (up to differences in machine precision) across machines given the seeds used here.

A number of anomalies in Table 4.2 should be addressed immediately. Several estimators have efficiencies greater than one implying that they performed better than the *asymptotically* optimal L-estimator. This is the case for the ordinary least squares (l_2) estimator in the Gaussian case. Here the optimal L-estimator is an untrimmed mean of the regression quantiles and suffers a slight deficiency (2%) relative to the classical least squares estimator. Perhaps more surprisingly the regression midmean (TRQ.25) outperforms the l_1-estimator for

Table 4.2

Finite Sample Efficiencies of Various
Estimators of the Bivariate Linear Model[1]

	Estimator						
Distribution	ARQ.05	ARQ.10	TRQ.10	TRQ.25	TRQ.5	l_2	l_∞
Normal	.96	.95	.96	.87	.65	1.02	.11
Cauchy	.89	.87	.47	.85	.84	.00	.00
Laplace	1.01	1.02	.90	1.04	1.00	.68	.03
Uniform	.32	.28	.26	.18	.12	.33	1.64
Exponential	.16	.15	.08	.08	.06	.06	.04
Lognormal	.27	.23	.09	.11	.09	.03	.02
Bimodal	2.30	2.09	.74	.42	.10	.64	.59

1. Sample size of the linear model is 100. All entries are reported relative to the "optimal L-estimator," e.g., mse $(\hat{\gamma}_{Lopt})$/mse $(\hat{\gamma}_{ARQ.05})$.

the Laplace (double-exponential) distribution, but again only rather slightly. In the uniform case, the optimal M-estimator is the l_∞-estimator and it substantially outperforms the optimal L-estimator. Here the rates of convergence are non-standard, so perhaps this disparity is not so surprising. More surprising is the poor performance of the "optimal" L-estimator in the bimodal mixture of normals. Here the optimal weight function looks like the "untrimmed regression quantile mean" except that the central quantiles are drastically downweighted. Clearly, the estimated J's deliver superior performance in this case, but the explanation is somewhat mysterious.

As the theory predicts, the adaptive L-estimators offer good performance over the entire range of distributions investigated. To our delight, they are particularly successful in the asymmetric and bimodal cases. But they offer high efficiency in the more familiar symmetric unimodal cases as well. Finally, we must emphasize that these results are based on a relatively small number of replications and very little experimentation with the smoothing methods employed to estimate the J functions. In future work we hope to report more extensive experimental results.

Appendix A

A Uniform Bahadur Representation for Regression Quantiles with Explicit Bounds.

Basically, the proof of Theorem 2.1 of Koenker and Portnoy (1986) will be followed exactly with bounds expressed explicitly as functions of the distribution and interval $(\alpha, 1-\alpha)$. However, this requires the result of Lemma 2.1 of Portnoy (1984) showing that $\|\hat{\gamma}\| = O_p(\frac{\log n}{n})^{1/2}$. To obtain explicit bounds, condition (2.10) of Portnoy (1984) must be replaced by condition X4 as described in Proposition 3.2 of Portnoy (1984) (with some modification of the argument). The conditions required here are the following.

A1: Conditions X1-X4 hold and the density f is continuous, bounded, and strictly positive.

A2: In addition to A1, the derivative f' exists and is uniformly bounded.

Note that for $F \in \mathrm{F}$, F_s satisfies A2; and, hence, Theorem A.1 below holds for $\hat{F}_n$ and $\hat{\gamma}$ defined by (1.6) and (1.3) (based on observations Y_i in (2.1)) and for F_s given by (2.3) for any $F \in \mathrm{F}$. Following the proofs of Lemma 2.1 and Proposition 2.2 (Portnoy, 1984) and keeping careful track of explicit bounds yields the folowing results:

Lemma A.1. Assume condition A1. *Then there exists* n_0 *and constants* $b_i(X)$ *depending only on the constants in conditions* X1-X4 *such that for* $n \geq n_0$.

$$\|\hat{\gamma}\| \leq K(X, f)\, R\, (\log n / n)^{1/2} \tag{A.1}$$

where

$$\begin{aligned} &P\{|R| \geq w\} \leq e^{-b_1(X)(w-1)^2 \log n} \qquad (\text{for } w \geq 2) \\ &K(X, f) = b_2(X)/\{\inf_{\alpha, b_3(X)} f(t)\}. \end{aligned} \tag{A.2}$$

Here, we define

$$\inf_{\alpha, b} f(t) = \inf\{f(t):\ F^{-1}(\alpha) - b \leq t \leq F^{-1}(1-\alpha) + b\}.$$

The results of Koenker and Portnoy (1986) can also be extended by providing firm bounds in terms of the density, f, and the constants in X1-X4. Again with $b_i(X)$ denoting constants (depending only on X1-X4), careful consideration of the proofs in Koenker and Portnoy (1986) yields the following results:

Lemma A.2. Assume condition A2 *and define for* $\delta \in R^p$ *and* $0 \leq \theta \leq 1$,

$$T(\delta, \theta) = \sum_{i=1}^{n} x_i \{I(u_i \leq F^{-1}(\theta) + x_i'\delta) - I(u_i \leq F^{-1}(\theta))\}$$
$$\tilde{T}(\delta, \theta) = T(\delta, \theta) - ET(\delta, \theta). \tag{A.3}$$

Then, for $\delta \in \Delta \equiv \{\delta: \|\delta\| \leq K(\log n/n)^{1/2}\}$ *and* $\alpha \leq \theta \leq 1 - \alpha$,

$$|ET(\delta, \theta) - nQ\delta f(F^{-1}(\theta))| \leq K(K + 1)b_1(X)\{\sup_x f(x) + \sup_x |f'(x)|\} \cdot n^{1/4}(\log n)^{1/2}$$

and

$$P\left\{\sup_{\delta \in \Delta, \alpha \leq \theta \leq 1-\alpha} \|\tilde{T}(\delta, \theta)\| \geq (n^{1/4} \log n)K^2 b_2(X)\{\sup_x f(x) + \sup_x |f'(x)|\}\right\}$$
$$\leq K \exp\{b_3(X) \sup_x f(x) - (\log n)\} + \frac{1}{\sqrt{n}}\{2\sup_x f(x)/\inf_\alpha f(t) + b_4(X)\}.$$

Combining Lemmas A.1 and A.2 yields,

Lemma A.3. Under A2,

$$P\left\{\sup_{\alpha \leq \theta \leq 1-\alpha} \|T((0, \hat{\gamma}), \theta)\| \geq n^{1/4}(\log n)B(X, F)\right\} \leq q(X, F)$$

where

$$B(X, F) = \frac{b_1(X)\{\sup_x f(x) + \sup_x |f'(x)|\}}{\{inf_{\alpha, b_2(X)} f(t)\}^2}$$

$$q(X, F) = \frac{b_3(X)}{\sqrt{n}} \exp\{b_4(X) \sup f(X)\}/\{\inf_{\alpha, b_5(X)} f(t)\}.$$

Lastly, as a consequence we have

Theorem A.1. Under condition A2, *with* $B(X, F)$ *and* $q(X, F)$ *defined above,*

$$\sup_{\alpha \le \theta \le 1-\alpha} |\hat{F}_n(F^{-1}(\theta)) - \frac{1}{n}\sum_{i=1}^{n} I(u_i \le F^{-1}(\theta))| \le n^{-3/4} (\log n)\, B(X, F)$$

and

$$\sup_{\alpha \le \theta \le 1-\alpha} \|\dot{Q}(\hat{\gamma}(\theta) - \gamma) f(F^{-1}(\theta)) - \frac{1}{n}\sum_{i=1}^{n} \dot{x}_i(\theta - I(u_i \le F^{-1}(\theta)))\| \le n^{-3/4} (\log n)\, B(X, F) + b_1(X)/n$$

except on a set with probability bounded above by $q(X, F)$.

REFERENCES

Bassett G., and Koenker, R. (1982) "An Empirical Quantile Function for the iid Linear Model," *JASA*, 77, 407-415.

Becker, R. A., and Chambers, J. M. (1984) *S: An Interactive Environment for Data Analysis and Graphics*, California: Wadsworth, Inc.

Beran, R. (1974) "Asymptotically Efficient Adaptive Rank Estimates in Location Models", *Annals of Statistics*, 2, 63-74.

Bickel, Peter J. (1982) "On Adaptive Estimation", *Annals of Statistics*, 10, 647-671.

Breiman, Leo (1968) *Probability*, Addison-Wesley, Reading, Mass.

Jureččková, J. and Sen, P. K. (1984) "On Adaptive Scale-Equivariant M-Estimators in Linear Models, *Statistics and Decisions*, 1, 31-46.

Cox, D. (1985) "A Penalty Method for Nonparametric Estimation of the Logarithmic Derivative of a Density Function", *Ann. Inst. Statist. Math.*, 37, 271-88.

De Jongh, P., and DeWet, T., (1986) Adaptive Trimmed Least Squares, preprint.

Hogg, R. (1980) On Adaptive Robust Inference, *Proceedings of the International Symposium on Statistics*, New York: North-Holland.

Koenker, Roger and Portnoy, Stephen (1986) "L-estimators for linear models", *JASA*, in press.

Koul, Hira (1969) "Asymptotic behavior of Wilcoxon type confidence regions in multiple linear regression", *Ann. Math. Stat.* 40, p. 1950.

Manski, Charles (1984) "Adaptive Estimation of Non-Linear Regression Models," *Econometric Reviews*, 3, 145-210.

Newey, W. (1987) Adaptive Estimation of Regression Models via Moment Restrictions, Princeton Research Memorandum No. 330.

Portnoy, Stephen (1984) "Tightness of the sequence of empirical c.d.f. processes defined from regression fractiles" in *Robust and Nonlin. Time Ser. Analysis* (eds. Franke, Härdle, Martin). Springer-Verlag, New York, p. 231.

Sacks, Jerome (1975) "An asymptotically efficient sequence of estimates of a location parameter," *Annals of Statistics*, 3, p. 285.

Silverman, B. (1986) *Density Estimation for Statistics and Data Analysis*, New York, Chapman Hall.

Stein, Charles (1956) "Efficient Nonparametric Testing and Estimation," *Proc.* 3rd *Berkeley Symp. Math. Statist. Prob.*, 1, 187-196.

Stone, Charles (1975) "Adaptive maximum likelihood estimators of a location parameter," *Annals of Statistics*, 3, p. 267.

Steven Portnoy
Department of Statistics
University of Illinois
Champaign, IL 61820

Roger Koenker
Department of Economics
University of Illinois
Champaign, IL 61820

HECKMAN
BINDERY INC.
JUN 95
ound-To-Please® N. MANCHESTER,
INDIANA 46962

UNIVERSITY OF ILLINOIS-URBANA
3 0112 060296032